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Asymptotic Approximation of the Solution of a Random Boundary Value Problem Containing Small White Noise

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This paper considers a nonlinear random differential equation

$$\begin{cases} d\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x})\, dt + \varepsilon \boldsymbol{\varphi}(t, \mathbf{x})\, d\mathbf{W}(t, \omega), \\ \mathbf{A}\mathbf{x}(0) + \mathbf{B}\mathbf{x}(1) = \boldsymbol{\alpha}(\omega), \end{cases} \qquad t \in [0, 1],$$

where $\boldsymbol{\alpha}(\omega)$ is $\mathscr{F}_1$-measurable and $\mathbf{w}$ is an R^m-valued Wiener process. By introducing a weak problem, the shooting method can be used to prove the uniqueness of the R^n-valued $\mathscr{F}_t$-measurable solution $\mathbf{x}(t)$ in the meaning of large probability. If the parameter ε is small, then $\mathbf{x}(t) = \mathbf{x}_0(t) + \varepsilon \mathbf{x}_1(t) + O(\varepsilon^2)$, where $\mathbf{x}_0(t)$ is the solution with $\varepsilon = 0$, and $\mathbf{x}_1(t)$ satisfies a linear random boundary value problem. For simplicity the discussion is given in the scalar case, but extensions to higher dimensions are readily apparent. © 1993 Academic Press, Inc.

1. INTRODUCTION

Because of the different nature of the underlying physical processes, the study of nonlinear boundary value problems is substantially more difficult than that of initial value problems. However, the fascinating field has developed significantly and a variety of techniques that are employed in the development of the theory are well recorded [1].

For the random problems, if they are studied through a sample calculus approach, then many methods, such as moments, upper and lower solutions, Green's function, and Prüfer substitution, have been used to derive some meaningful facts [2–5, 7]. But when white noise is contained in the problem, only a relatively few results are obtained. One of the reasons for the lack of development of the theory in this case is perhaps due to the fact that the $\mathscr{F}_t$-measurability, which is necessary for the definition of the Ito integral, is not always satisfied.

To overcome this difficulty, one may use the extended Ito integral to find some interesting properties for the random eigenvalue problems or the Helmholts equation, but the solutions are not $\mathscr{F}_t$-measurable [6, 8]. If the

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equation considered has a special form $dx/dt + Ax + Bf(x) = B\,dW/dt$, and the integral is defined by integration by parts, then under some assumptions the corresponding boundary value problem has a unique solution, and if we limit the dimension to be one and the function $f(x)$ to be affine then this solution must be a Markov field [13].

Now we try to find the possible $\mathscr{F}_t$-measurable solution under the meaning of the Ito type integral for the general problem

$$\begin{cases} dx(t) = f(t, x(t))\,dt + \varphi(t, x(t))\,dW(t, w), \\ Ax(0) + Bx(1) = \alpha(\omega), \end{cases} \qquad t \in T \triangleq [0, 1], \quad (1.1)$$

where $W(t, \omega)$ is the Wiener process. If its solution $x(t)$ exists, then $x(0)$ must be $\mathscr{F}_0$-measurable and $\alpha(\omega)$ must be $\mathscr{F}_1$-measurable. Since we know that the shooting method is one of the powerful methods both for deterministic and random boundary value problems, and that when using it the possible initial value must be derived from some known information including the boundary value α, then its measurability must be considered carefully.

To avoid the possible obstacle of selecting an $\mathscr{F}_0$-measurable initial value, it is helpful to introduce a weak problem

$$\begin{cases} dx(t) = f(t, x(t))\,dt + \varphi(t, x(t))\,dW(t, \omega), \\ E[Ax(0) + Bx(1) - \alpha(\omega) \mid \mathscr{F}_0] = 0, \end{cases} \qquad t \in T, \qquad (1.2)$$

where $E[\cdot \mid \mathscr{F}_0]$ denotes the conditional expectation. If the $\mathscr{F}_t$-measurable solution of (1.2) exists, then we say that it is the weak solution of (1.1). It is easy to find that every solution of (1.1) must be the weak solution, and if the weak solution satisfies the condition that $Ax(0) + Bx(1) - \alpha(\omega)$ is $\mathscr{F}_0$-measurable, then it must be the solution of (1.1).

In the linear case, if $f(t, x) = A^*(t)\,x + a(t)$, $\varphi(t, x) = B^*(t)\,x + b(t)$, then the weak solution exists uniquely (see [11], for example), with the expression

$$\begin{aligned} x(t) &= \Phi(t)\left\{x(0) + \int_0^t \Phi^{-1}(t)[a(t) - B^*(t)\,b(t)]\,dt \right. \\ &\qquad \left. + \int_0^t \Phi^{-1}(t)\,b(t)\,dW(t, \omega)\right\}, \\ \Phi(t) &= \exp\left\{\int_0^t [A^*(t) - B^{*2}(t)/2]\,dt + \int_0^t B^*(t)\,dW(t, \omega)\right\}, \\ X(0) &= [A + BE(\Phi(1) \mid \mathscr{F}_0)]^{-1} \\ &\qquad \times \left\{E(\alpha(\omega) \mid \mathscr{F}_0) - BE\left[\Phi(1)\int_0^1 \Phi^{-1}(a - B^*b)\,dt \right.\right. \\ &\qquad \left.\left. + \Phi(1)\int_0^1 \Phi^{-1} b\,dW(t, \omega) \mid \mathscr{F}_0\right]\right\}. \end{aligned} \qquad (1.3)$$

For the nonlinear problem, if we notice that equations for which the unique solution exists are very limited even for deterministic boundary value problems, then instead of the usual uniqueness the so-called "local uniqueness" will be used in our main results, and a small parameter ε will be introduced in our problem

$$\begin{cases} dx(t) = f(t, x(t))\,dt + \varepsilon\varphi(t, x(t))\,dW(t, \omega), \\ Ax(0) + Bx(1) = \alpha(\omega), \end{cases} \qquad t \in T. \tag{1.4}$$

If we set $\alpha_0(\omega) = E[\alpha \mid \mathscr{F}_0]$, then the weak problem corresponding to (1.2) becomes

$$\begin{cases} dx(t) = f(t, x(t))\,dt + \varepsilon\varphi(t, x(t))\,dW(t, \omega), \\ E[Ax(0) + Bx(1) \mid \mathscr{F}_0] = \alpha_0(\omega), \end{cases} \qquad t \in T, \tag{1.5}$$

which is the main problem to be studied in this paper.

Although we don't know whether the solution of (1.4) exists in the general case, we can prove in our paper that the solution of (1.5) exists uniquely in some sense, and that it can be expanded into $x(t) = x_0(t) + \varepsilon x_1(t) + o(\varepsilon^2)$. Thus if the solution of (1.4) exists, it must be unique locally and have the expansion shown above, where $x_0(t)$ is the solution of (1.5) with $\varepsilon = 0$, and $x_1(t)$ is the solution of a linear random boundary value problem (4.1). These ideas are illustrated by two examples in the last section, and some sufficient conditions for the main results are also pointed out at the end of this paper.

2. Random Mapping $T(\delta)$ and ω Set Ω_{δ^*}

For simplicity we will omit t or ω sometimes, and use $\cdot \in \mathscr{F}_t$ to denote its $\mathscr{F}_t$-measurability.

In order to construct our solution of (1.5), it is helpful to introduce a random mapping $T(\delta)$, and to specify an ω set Ω_{δ^*}, which has large probability for small ε.

We first suppose that $x_0(t) \in \mathscr{F}_t$ is the solution of (1.5) with $\varepsilon = 0$. If we notice that $x_0(0)$ is $\mathscr{F}_0$-measurable, and when $\varepsilon = 0$, $\alpha_0(\omega)$ is the only random factor in our problem, then $x_0(t)$ can be assumed to be $\mathscr{F}_0$-measurable for all $t \in T$, and the corresponding equation satisfied becomes

$$dx_0(t) = f(t, x_0)\,dt, \qquad Ax_0(0) + Bx_0(1) = \alpha_0, \quad t \in T. \tag{2.1}$$

If it has the property that

$$|J| \triangleq \left| A + B \exp \int_0^1 f_x(t, x_0)\,dt \right| \geqslant K_1 > 0, \tag{2.2}$$

where K_1 is a deterministic constant, then for any random variable $\delta \in \mathscr{F}_0$ we can define a random mapping

$$T(\delta) \triangleq \delta - J^{-1}E[A\delta + By(1, \delta) \mid \mathscr{F}_0], \tag{2.3}$$

where $y(t, \delta) \in \mathscr{F}_t$ is the unique solution of the following Ito type random initial value problem:

$$\begin{aligned} dy(t) &= [f(t, x_0 + y) - f(t, x_0)]\, dt \\ &\quad + \varepsilon\varphi(t, x_0 + y)\, dW(t), \qquad y(0) = \delta,\ t \in T. \end{aligned} \tag{2.4}$$

Now we try to find a way to show that $T(\delta)$ is a contracting random mapping defined on a complete metric space, then by the well known theorem, there exists a unique random fixed point $\hat{\delta}(\omega)$, such that $T(\hat{\delta}) = \hat{\delta}$. So we can obtain the weak solution of (1.4) by setting $x(t) = x_0(t) + y(t, \hat{\delta})$. Although this argument is not always valid for all $\omega \in \Omega$, we have proved that in some special cases, it is true for some $\omega \in \mathscr{A}_\Delta^\varepsilon$ by showing that for "all" random variables δ with $|\delta| \leqslant \Delta$, T is contracting on $\mathscr{A}_\Delta^\varepsilon$ [7–10]. However, for our nonlinear problem the above assertion is not valid for "all" δ. But if our main intrest is in the existence of a solution of (1.5), and if we can prove that the contracting property of $T(\delta)$ is valid for a specially selected sequence $\delta_n(\omega)$, $n = 0, 1, \ldots$, then the limit of $\delta_n(\omega)$ still exists and the corresponding solution $x(t) = x_0(t) + y(t, \hat{\delta})$ can still be constructed. To do so, we define $\delta_n(\omega)$, $n = 0, 1, \ldots$, by the formula

$$\delta_{n+1}(\omega) = L(\delta_n) \triangleq \delta_n - I_n J^{-1}E[A\delta_n + By(1, \delta_n) \mid \mathscr{F}_0], \tag{2.5}$$

where $\delta_0 \in \mathscr{F}_0$ is a fixed random variable with $|\delta_0| \leqslant \Delta$ (a.s.), and I_n is the indicator of the ω set $\Omega_{\delta_n}(\Delta) \triangleq \{\omega \mid |T(\delta_n)| \leqslant \Delta\}$.

LEMMA 1. *For problem* (1.5), *we suppose that*

(1) *for all $t \in T$, the σ-algebras of events $\mathscr{F}_t$ are defined so as to possess the properties*

> *$W(t)$ is an $\mathscr{F}_t$-measurable Wiener process with independent increments; for $0 \leqslant t_1 < t_2 \leqslant 1$, $\mathscr{F}_{t_1} \subset \mathscr{F}_{t_2}$,*

(2) *$\alpha(\omega)$ is an $\mathscr{F}_1$-measurable random variable with $E\,|\alpha^4(\omega)| < \infty$;*

(3) *$f(t, x)$, $\varphi(t, x)$, $f_x(t, x)$, $\varphi_x(t, x)$, $f_{xx}(t, x)$, $\varphi_{xx}(t, x)$ are continuous functions defined on $[0, 1] \times (-\infty, +\infty)$, and have the properties*

$$\begin{aligned} |f_x(t, x)| + |\varphi_x(t, x)| &\leqslant K_2, \\ |f_{xx}(t, x)| + |\varphi_{xx}(t, x)| &\leqslant K_3, \\ |f(t, x)| + |\varphi(t, x)| &\leqslant K_4(1 + |x|); \end{aligned} \tag{2.6}$$

(4) *the* $\mathscr{F}_0$*-measurable solution* $x_0(t)$ *exists (may not be unique), which satisfies Eq.* (2.1), *and has properties (see Appendix for sufficient conditions)*

$$E[\sup_T x_0^4(t)] < \infty$$

$$|J| \triangleq \left| A + B \exp \int_0^1 f_x(t, x_0)\, dt \right| \geqslant K_1 > 0,$$

where K_i, $i = 1, 2, \ldots$, *denote some deterministic positive constants.*

Then there exists two constants $K > 0$, $\Delta^* > 0$, *independent on* ε *and the choices of* $\delta(\omega)$, *such that for any* $\delta(\omega) \in \mathscr{F}_0$ *with* $|\delta| \leqslant \Delta^*$ (a.s.), *the* ω *set*

$$\Omega_\delta(\Delta^*) \triangleq \{\omega \mid |T(\delta)| \leqslant \Delta^*\}$$

is $\mathscr{F}_0$*-measurable with large probability*

$$\operatorname{prob}\{\Omega_\delta(\Delta^*)\} \geqslant 1 - K\varepsilon^2$$

for small ε.

Proof. The first assertion can be obtained directly from the fact that $T(\delta)$ is $\mathscr{F}_0$-measurable. To prove the second one, we need the following notations for simplicity

$$y_i = y(t, \delta_i), \qquad \varphi_i = \varphi(t, x_0 + y_i), \qquad f_i = f(t, x_0 + y_i)$$
$$f_x(\lambda) = f_x(t, x_0 + y_1 + \lambda(y_2 - y_1)),$$
$$\varphi_x(\lambda) = \varphi_x(t, x_0 + y_1 + \lambda(y_2 - y_1)), \qquad 0 \leqslant \lambda \leqslant 1.$$

Furthermore we will use $\xi_i(t)$, $\eta_i(t)$ to denote some $\mathscr{F}_t$-measurable functions with the property that for every $i = 1, 2, \ldots$, there is a deterministic constant independent on ε and δ such that

$$E \sup_T \xi_i^4(t) + \sup_T |\eta_i(t)| \leqslant \text{const.}$$

Now we establish some different expressions for y_i, $i = 1, 2$, and their difference, which will be used later. From (2.4) we know that $y_2 - y_1$ satisfies a linear equation

$$\begin{aligned} d(y_2 - y_1) &= (f_2 - f_1)\, dt + \varepsilon(\varphi_2 - \varphi_1)\, dW \\ &= f_x(\lambda)(y_2 - y_1)\, dt + \varepsilon(\varphi_2 - \varphi_1)\, dW \qquad (2.7) \\ y_2 - y_1|_{t=0} &= \delta_2 - \delta_1. \end{aligned}$$

Since $f_x(\lambda)$ is $\mathscr{F}_t$-measurable, the above equation is of the Ito type, and its unique solution can be obtained from (1.3):

$$y_2 - y_1 = e^{\int_0^t f_x(\lambda)\,dt}\left\{(\delta_2 - \delta_1) + \int_0^t e^{-\int_0^t f_x(\lambda)\,dt}(\varphi_2 - \varphi_1)\,dW(t)\right\}. \tag{2.8}$$

If we rewrite (2.7) in the form

$$\begin{aligned} &d(y_2 - y_1) = f_x(\lambda)(y_2 - y_1)\,dt + \varepsilon\varphi_x(\lambda)(y_2 - y_1)\,dW(t) \\ &y_2 - y_1|_{t=0} = \delta_2 - \delta_1 \end{aligned} \tag{2.9}$$

and use (1.3) again, then

$$y_2 - y_1 = (\delta_2 - \delta_1)\exp\left\{\int_0^t [f_x(\lambda) - \varepsilon^2\varphi_x^2(\lambda)/2]\,dt + \varepsilon\int_0^t \varphi_x(\lambda)\,dW(t)\right\}, \tag{2.10}$$

where the λ's in φ_x and f_x may be different, but we still use the same notation for simplicity.

To estimate $y_2 - y_1$, we denote the term $\exp\{\ldots\}$ in (2.10) by $e(t)$. It is $\mathscr{F}_t$-measurable and satisfies

$$de(t) = f_x(\lambda)\,e(t)\,dt + \varepsilon\varphi_x(\lambda)\,e(t)\,dW(t), \qquad e(0) = 1. \tag{2.11}$$

Then the fact that $E\sup_T e^4(t) < \infty$ gives the estimation of $y_2 - y_1$ by substituting $\varphi_2 - \varphi_1 = \varphi_x(\lambda)(y_2 - y_1) = \varphi_x(\lambda)(\delta_2 - \delta_1)\,e(t)$ into (2.8),

$$\begin{aligned} y_2 - y_1 &= (\delta_2 - \delta_1)\left\{e^{\int_0^t f_x(\lambda)\,dt} + \varepsilon e^{\int_0^t f_x(\lambda)\,dt}\int_0^t e^{-\int_0^t f_x(\lambda)\,dt}\varphi_x(\lambda)\,e(t)\,dW(t)\right\} \\ &\triangleq (\delta_2 - \delta_1)\{e^{\int_0^t f_x(\lambda)\,dt} + \varepsilon\xi_1(t)\} &(2.12) \\ &\triangleq (\delta_2 - \delta_1)\{\eta_1(t) + \varepsilon\xi_1(t)\}, &(2.13) \end{aligned}$$

where ξ_1 and η_1 are introduced for simplicity with the properties mentioned above.

For y_1, we have some similar results:

$$\begin{aligned} y_1 &= e^{\int_0^t f_x(\lambda)\,dt}\left\{\delta_1 + \varepsilon\int_0^t e^{-\int_0^t f_x(\lambda)\,dt}\varphi_1\,dW\right\} &(2.14) \\ &\triangleq \delta_1\,e^{\int_0^t f_x(\lambda)\,dt} + \varepsilon\xi_2(t) &(2.15) \\ &\triangleq \delta_1\eta_2(t) + \varepsilon\xi_2(t). &(2.16) \end{aligned}$$

Now we are in a position to consider the probability of $\Omega_\delta(\Delta^*)$. Since

$$\begin{aligned} T(\delta) &= \delta - J^{-1}E[A\delta + By(1,\delta) \mid \mathscr{F}_0] \\ &= J^{-1}B\{E[\delta e^{\int_0^1 f_x(t,x_0)\,dt} - y(1,\delta) + y(1,0) \mid \mathscr{F}_0] - E[y(1,0) \mid \mathscr{F}_0]\}, \end{aligned} \tag{2.17}$$

and (2.12) gives

$$\begin{aligned} &y(t,\delta) - y(t,0) - \delta e^{\int_0^t f_x(t,x_0)\,dt} \\ &\quad = \delta[e^{\int_0^t f_x(\lambda)\,dt} - e^{\int_0^t f_x(t,x_0)\,dt}] + \varepsilon\delta\xi_1(t) \\ &\quad = \delta \exp\left[\int_0^t (f_x(t,x_0) + \lambda(f_x(\lambda) - f_x(t,x_0)))\,dt\right] \\ &\qquad \cdot \int_0^t f_{xx}(\lambda)[y(t,0) + \lambda(y(t,\delta) - y(t,0))]\,dt + \varepsilon\delta\xi_1(t), \end{aligned}$$

we can use (2.13) and (2.16) to find functions ξ_3, η_3, and a constant $K_5 > 0$, independent on ε, such that

$$\begin{aligned} |T(\delta)| &\leqslant |BJ^{-1}| \cdot E\left\{|\delta| \int_0^1 |\eta_3(t)| \cdot |y(t,0) + \lambda(y(t,\delta) - y(t,0))|\,dt \right. \\ &\qquad \left. + \varepsilon\,|\delta\xi_1(1)| + \varepsilon\,|\xi_2(1)| \mid \mathscr{F}_0\right\} \\ &\leqslant K_5\{\Delta^{*2} + \Delta^*\varepsilon E[\sup_T |\xi_3(t)| \mid \mathscr{F}_0] + \Delta^{*2}\varepsilon E[\sup_T |\xi_1(t)| \mid \mathscr{F}_0] \\ &\qquad + \varepsilon E[|\xi_2(1)| \mid \mathscr{F}_0]\}. \end{aligned}$$

If a fixed number $\Delta^* \in (0, \min(1/2K_5, 1))$ is selected, then for $\omega \notin \Omega_\delta(\Delta^*)$ we can have

$$\Delta^* < |T(\delta)| \leqslant \frac{\Delta^*}{2} + \frac{\varepsilon}{2} E[\sup_T |\xi_4(t)| \mid \mathscr{F}_0],$$

or

$$\Delta^* < \varepsilon E[\sup_T |\xi_4(t)| \mid \mathscr{F}_0].$$

Hence, the lemma can be proved immediately by the Tchebichev inequality

$$\begin{aligned} \mathrm{Prob}\{\Omega_\delta(\Delta^*)\} &= 1 - \mathrm{Prob}\{\Omega_\delta^c(\Delta^*)\} \\ &\geqslant 1 - \mathrm{Prob}\{\omega \mid \Delta^* < \varepsilon E[\sup_T |\xi_4(t)| \mid \mathscr{F}_0]\} \\ &\geqslant 1 - \varepsilon^2 E[\sup_T |\xi_4(t)|^2]/\Delta^{*2} \triangleq 1 - K\varepsilon^2, \end{aligned}$$

where K is a deterministic constant independent on ε and the choices of δ for fixed Δ^*.

LEMMA 2. *Under the assumptions of Lemma* 1, *for any fixed* $\delta_0(\omega) \in \mathscr{F}_0$, *if* $|\delta_0(\omega)| \leqslant \Delta^*$ *(a.s.), then we can find an* $\mathscr{F}_0$*-measurable* ω *set* Ω_{δ^*} *contained in* $\Omega_{\delta_n}(\Delta^*)$ *for all* $n = 0, 1, 2, ...,$ *such that* $\mathrm{Prob}\{\Omega_{\delta^*}\} \geqslant 1 - K\varepsilon^2$.

Proof. From (2.5) we know

$$\delta_{n+1} = L(\delta_n) = \begin{cases} T(\delta_n), & \omega \in \Omega_{\delta_n}(\Delta^*) \\ \delta_n, & \omega \notin \Omega_{\delta_n}(\Delta^*) \end{cases}$$

and then, by induction, $|\delta_{n+1}| \leqslant \Delta^*$ (a.s.) and $\delta_{n+1} \in \mathscr{F}_0$.

When $\omega \notin \Omega_{\delta_n}(\Delta^*)$, the fact $\delta_n = \delta_{n+1}$ tells us $y(1, \delta_{n+1}) = y(1, \delta_n)$ from (2.10). Then the $\mathscr{F}_0$-measurability of $\Omega_{\delta_n}(\Delta^*)$ yields $|T(\delta_{n+1})| = |T(\delta_n)| > \Delta^*$, and $\Omega_{\delta_{n+1}}(\Delta^*) \subset \Omega_{\delta_n}(\Delta^*)$. So we can find an $\mathscr{F}_0$-measurable ω set $\Omega_{\delta^*} = \lim_{n \to \infty} \Omega_{\delta_n}(\Delta^*) = \bigcap_n \Omega_{\delta_n}(\Delta^*)$ which is contained in $\Omega_{\delta_n}(\Delta^*)$ for all $n = 0, 1, 2, ...,$ and has the probability estimate

$$\mathrm{Prob}\{\Omega_{\delta^*}\} = \lim_{n \to \infty} \mathrm{Prob}\{\Omega_{\delta_n}(\Delta^*)\} \geqslant 1 - K\varepsilon^2,$$

where Lemma 1 has been used again to derive the last inequality, and then it terminates our proof.

3. EXISTENCE AND LOCAL UNIQUENESS OF THE SOLUTION

To prove the existence of the solution, we limit

$$\omega \in \Omega_{\hat{\delta}} \triangleq \Omega_{\delta^*} \cap \{\omega \mid \varepsilon \sup_T |X_0(t)| \leqslant l\},$$

where l is a fixed deterministic positive number specified later. It is easily seen that $\Omega_{\hat{\delta}} \in \mathscr{F}_0$ with $P\{\Omega_{\hat{\delta}}\} \geqslant 1 - \hat{K}\varepsilon^2$ and that the limitation of ω can be realized by introducing a new mapping

$$S(\delta_n) \triangleq I_{\hat{\delta}} L(\delta_n) = I_{\hat{\delta}} T(\delta_n) = I_{\hat{\delta}}\{\delta_n - J^{-1} E[A\delta_n + By(1, \delta_n) \mid \mathscr{F}_0]\},$$

where $I_{\hat{\delta}}$ is the indicator of $\Omega_{\hat{\delta}}$.

This mapping is contracting for δ_n, $n = 0, 1, ...$ In fact we can have

THEOREM 1. *Under the assumptions of Lemma* 1, *there exist two small positive constants* $\hat{\Delta}, \varepsilon_0$, *such that for any fixed* $\mathscr{F}_0$*-measurable random* *variable* $\delta_0(\omega)$, *if* $|\delta_0| \leqslant \hat{\Delta}$ *(a.s.),* $0 \leqslant \varepsilon \leqslant \varepsilon_0 \leqslant 1$, *the above random mapping* S *is a contracting mapping for* δ_n, *that is,*

$$\begin{aligned} &(1) \qquad S(\delta_n) \in \mathscr{F}_0,\ |S(\delta_n)| \leqslant \hat{\Delta}\ (a.s.), \\ &(2) \qquad E[(S(\delta_n) - S(\delta_m))^{2j}] \leqslant 2^{-2j} E[I_{\hat{\delta}}(\delta_n - \delta_m)^{2j}] \\ &\qquad\qquad \leqslant 2^{-2j} E[(\delta_n - \delta_m)^{2j}], \end{aligned}$$

where $j = 1, 2$, *and* $n, m = 0, 1, \ldots$.

Proof. The first assertion follows immediately from the definitions of $S(\delta_n)$ and $I_{\hat{\delta}}$. To prove the second one, we rewrite $T(\delta_n) - T(\delta_m)$ in the form

$$T(\delta_n) - T(\delta_m) = BJ^{-1} E\{(\delta_n - \delta_m)\, e^{\int_0^1 f_x(t, x_0)\, dt} - (y_n(1) - y_m(1)) \mid \mathscr{F}_0\}, \tag{3.1}$$

and investigate properties of the term $y_{nm}(t) = y_n(t) - y_m(t) - (\delta_n - \delta_m)\, e^{\int_0^t f_x(t, x_0)\, dt}$. Since it satisfies a linear equation

$$\begin{aligned} dy_{nm}(t) &= f_x(t, x_0)\, y_{nm}\, dt \\ &\quad + [f_x(\lambda) - f_x(t, x_0)](y_n - y_m)\, dt + \varepsilon \varphi_x(\lambda)(y_n - y_m)\, dW(t) \\ y_{nm}(0) &= 0 \end{aligned}$$

and its solution can be evaluated from (1.3):

$$\begin{aligned} y_{nm}(t) = e^{\int_0^t f_x(t, x_0)\, dt} \Big\{ \int_0^t & e^{-\int_0^t f_x(t, x_0)\, dt} [(f_x(\lambda) - f_x(t, x_0))(y_n - y_m)\, dt \\ & + \varepsilon \varphi_x(\lambda)(y_n - y_m)\, dW(t)] \Big\}. \end{aligned}$$

Substituting from (2.8) into it, we have

$$\begin{aligned} y_{nm}(t) = e^{\int_0^t f_x(t, x_0)\, dt} \Big\{ & \int_0^t e^{\int_0^t (f_x(\lambda) - f_x(t, x_0))\, dt} [(f_x(\lambda) - f_x(t, x_0)](\delta_n - \delta_m)\, dt \\ & + \varepsilon \int_0^t e^{-\int_0^t f_x(t, x_0)\, dt} \varphi_x(\lambda)(y_n - y_m)\, dW \\ & + \varepsilon \int_0^t e^{\int_0^t [f_x(\lambda) - f_x(t, x_0)]\, dt} [f_x(\lambda) - f_x(t, x_0)] \\ & \cdot \left[\int_0^t e^{-\int_0^t f_x(\lambda)\, dt} (\varphi_n - \varphi_m)\, dW \right] dt \Big\}. \end{aligned}$$

If notations ξ_i and η_i are introduced again (see Lemma 1), then it becomes

$$|y_{nm}(t)| \leqslant e^{\int_0^t f_x(t, x_0)\,dt} \left\{ \left| \int_0^t e^{\int_0^t [f_x(\lambda) - f_x(t, x_0)]\,dt} f_{xx}(\lambda)\, y_m(\delta_n - \delta_m)\, dt \right| + \eta_4(t)(\delta_n - \delta_m)^2 + \varepsilon \left| \int_0^t \eta_5(t)(y_n - y_m)\, dW \right| + \varepsilon \left| \int_0^t \eta_6(t) \left[\int_0^t \eta_7(t)(y_n - y_m)\, dW \right] dt \right| \right\}. \tag{3.2}$$

The function $y_m(\delta_n - \delta_m)$ in the first term satisfies

$$dy_m(t)(\delta_n - \delta_m) = f_x(\lambda)\, y_m(\delta_n - \delta_m)\, dt + \varepsilon\varphi(t, x_0 + y_m)(\delta_n - \delta_m)\, dW$$

$$y_m(0)(\delta_n - \delta_m) = \delta_m(\delta_n - \delta_m).$$

Thus

$$[y_m(t)(\delta_n - \delta_m)]^2 \leqslant 3 \left\{ \delta_m^2(\delta_n - \delta_m)^2 + \int_0^t f_x^2(\lambda)\, y_m^2(\delta_n - \delta_m)^2\, dt + \varepsilon^2 \left[\int_0^t \varphi(t, x_0 + y_m)(\delta_n - \delta_m)\, dW \right]^2 \right\},$$

and then the $\mathscr{F}_0$-measurability of Ω_δ yields

$$\begin{aligned} &E[I_\delta y_m^2(t)(\delta_n - \delta_m)^2] \\ &\leqslant 3 \left\{ E[I_\delta \delta_m^2(\delta_n - \delta_m)^2] + K_2^2 \int_0^t E[I_\delta y_m^2(\delta_n - \delta_m)^2]\, dt + 2\varepsilon^2 K_4^2 \int_0^t E[I_\delta(\delta_n - \delta_m)^2\, (1 + 2x_0^2 + 2y_m^2)]\, dt \right\} \\ &\leqslant 3E[I_\delta \delta_m^2(\delta_m - \delta_n)^2] + 6K_4^2\varepsilon^2 E[I_\delta(\delta_n - \delta_m)^2] \\ &\quad + K_6 \int_0^t E[I_\delta y_m^2(\delta_n - \delta_m)^2]\, dt + 12K_4^2 E[I_\delta(\delta_n - \delta_m)^2\, \varepsilon^2 \sup_T x_0^2(t)]. \end{aligned}$$

So we have

$$\begin{aligned} &E[I_\delta y_m^2(t)(\delta_n - \delta_m)^2] \\ &\leqslant 3E[I_\delta \delta_m^2(\delta_n - \delta_m)^2] + 6(K_4^2\varepsilon^2 + 2K_4^2 l^2)\, E[I_\delta(\delta_n - \delta_m)^2] \\ &\quad + K_6 \int_0^t E[I_\delta y_m^2(\delta_n - \delta_m)^2]\, dt \\ &\leqslant K_7 \{ E[I_\delta \delta_m^2(\delta_n - \delta_m)^2] + (\varepsilon^2 + l^2)\, E[I_\delta(\delta_n - \delta_m)^2] \}, \end{aligned} \tag{3.3}$$

where Gronwall's inequality has been used to derive the last inequality.

The last two terms in (3.2) can be estimated by an equation similar to (2.9); that is,

$$E[I_{\hat\delta} \sup_T (y_n(t) - y_m(t))^2] \leqslant K_8 E[I_{\hat\delta}(\delta_n - \delta_m)^2]. \tag{3.4}$$

From this and (3.1), (3.2), (3.3), we then obtain the estimate

$$\begin{aligned} E[S(\delta_n) - S(\delta_n)]^2 \leqslant K_9\{ & E[I_{\hat\delta}(\delta_n - \delta_m)^4] + E[I_{\hat\delta}\delta_m^2(\delta_n - \delta_m)^2] \\ & + (\varepsilon^2 + l^2)\, E[I_{\hat\delta}(\delta_n - \delta_m)^2]\}. \end{aligned}$$

If we select $0 \leqslant \varepsilon \leqslant \varepsilon_0 \triangleq \sqrt{1/12K_9}$, $0 < \hat\Delta \leqslant \min\{\Delta^*, \sqrt{1/60K_9}\}$ and $0 < l \leqslant \sqrt{1/12K_9}$, then

$$E[(S(\delta_n) - S(\delta_m))^2] \leqslant \tfrac{1}{4} E[I_{\hat\delta}(\delta_n - \delta_m)^2]. \tag{3.5}$$

This is the second assertion for $j = 1$; in the case of $j = 2$ we can prove it by selecting other suitable ε, $\hat\Delta$, and l.

THEOREM 2. *Under the assumptions of Theorem* 1, *near every* $x_0(t)$ *there exists one and only one solution of* (1.5) *in the sense of large probability.*

Proof. From Theorem 1 and (2.5), we know that $E[S(\delta_n) - S(\delta_m)]^2 = E[I_{\hat\delta}(\delta_{n+1} - \delta_{m+1})^2] \leqslant (1/4)\, E[I_{\hat\delta}(\delta_n - \delta_m)^2]$. Then $\{I_{\hat\delta}\delta_n\}$ is a fundamental sequence, and it has a limit $\hat\delta$ in the meaning of a mean square. If we notice that $\{\omega \mid |\hat\delta| > \hat\Delta\} = \bigcup_{K=1}^{\infty} \{\omega \mid |\hat\delta| > \hat\Delta + 1/K\}$, and that $\{\omega \mid |\hat\delta| > \hat\Delta + 1/K\} \subset \{\omega \mid |\hat\delta - I_{\hat\delta}\delta_n| + |I_{\hat\delta}\delta_n| > \hat\Delta + 1/K\} = \{\omega \mid |\hat\delta - I_{\hat\delta}\delta_n| > \hat\Delta + 1/K - |I_{\hat\delta}\delta_n|\} \cap \{\omega \mid |I_{\hat\delta}\delta_n| \leqslant \hat\Delta\} \subset \{\omega \mid |\hat\delta - I_{\hat\delta}\delta_n| > 1/K\}$ then the fact that $I_{\hat\delta}\delta_n \to \hat\delta$ leads to the results that the sets $\{\omega \mid |\hat\delta| > \hat\Delta + 1/K\}$, $k = 1, 2, \ldots$, and $\{\omega \mid |\hat\delta| > \hat\Delta\}$ are null sets with probability 0. So that $|\hat\delta| \leqslant \hat\Delta$ (a.s.). Now we consider $S(\hat\delta)$. Since

$$S(\hat\delta) - S(\delta_n) = I_{\hat\delta}\{(\hat\delta - \delta_n) - J^{-1}E[A(\hat\delta - \delta_n) + B(y(1, \hat\delta) - y(1, \delta_n)) \mid \mathscr{F}_0]\},$$

and $I_{\hat\delta} \in \mathscr{F}_0$, we can use (2.9) and Gronwall's inequality to obtain the estimate $E[S(\hat\delta) - S(\delta_n)]^2 \leqslant K_{10} E[I_{\hat\delta}(\hat\delta - \delta_n)^2]$. Then as $n \to \infty$, $S(\delta_n) \to S(\hat\delta)$ in the meaning of mean square, which means $\hat\delta = S(\hat\delta)$ by noticing $S(\delta_n) = I_{\hat\delta}\delta_{n+1} \to \hat\delta$ and the uniqueness of the limit.

The function $x(t) \triangleq x_0(t) + y(t, \hat\delta)$ is obviously the solution of (1.5) when $\omega \in \Omega_{\hat\delta}$. Similar to $y(t, \hat\delta)$, if we have another function $y(t, \hat\zeta)$ such that $x_0(t) + y(t, \hat\zeta)$ is the solution on an ω set $\Omega_{\hat\zeta}$ with $\mathrm{Prob}\{\Omega_{\hat\zeta}\} \geqslant 1 - \hat K\varepsilon^2$, and $\hat\zeta$ has the properties $y(0, \hat\zeta) = \hat\zeta \in \mathscr{F}_0$, $|\hat\zeta| \leqslant \hat\Delta$ (a.s.), $\hat\zeta = S(\hat\zeta) = I_{\hat\zeta}T(\hat\zeta)$, then on ω set $\Omega_{\hat\delta} \cap \Omega_{\hat\zeta}$, we can have an estimate similar to (3.5),

$$\begin{aligned} E\{[S(\hat\delta) - S(\hat\zeta)]^2 I(\Omega_{\hat\delta} \cap \Omega_{\hat\zeta})\} &= E\{I(\Omega_{\hat\delta} \cap \Omega_{\hat\zeta})[T(\hat\delta) - T(\hat\zeta)]^2\} \\ &\leqslant \tfrac{1}{4} E\{I(\Omega_{\hat\delta} \cap \Omega_{\hat\zeta})(\hat\delta - \hat\zeta)^2\}, \end{aligned}$$

which yields from the property of a fixed point that

$$E\{I(\Omega_{\hat\delta}\cap\Omega_{\hat\zeta})(\hat\delta-\hat\zeta)^2\}\leqslant\tfrac{1}{4}E\{I(\Omega_{\hat\delta}\cap\Omega_{\hat\zeta})(\hat\delta-\hat\zeta)^2\},$$

or

$$E\{I(\Omega_{\hat\delta}\cap\Omega_{\hat\zeta})(\hat\delta-\hat\zeta)^2\}=0.$$

This implies $I(\Omega_{\hat\delta}\cap\Omega_{\hat\zeta})(\hat\delta-\hat\zeta)=0$ (a.s.). Then we can extend $y(t,\hat\zeta)$ from $\Omega_{\hat\zeta}$ to $\Omega_{\hat\delta}\cup\Omega_{\hat\zeta}$, by using (2.10) and defining the new initial value

$$\zeta=\begin{cases}\hat\zeta, & \omega\notin\Omega_{\hat\delta}\\ \hat\delta, & \omega\in\Omega_{\hat\delta}.\end{cases}$$

Since on $\Omega_{\hat\delta}$, $y(t,\zeta)=y(t,\hat\delta)$ and on $\Omega_{\hat\zeta}$, $y(t,\zeta)=y(t,\hat\zeta)$, then we can say that near every fixed $x_0(t)$ there exists one and only one solution of (1.5) on the ω set $\Omega_{\hat\delta}$, which has large probability when ε is small enough.

4. The Approximate Solutions

Suppose $x_1(t)$ is the $\mathscr{F}_t$-measurable solution of the linear equation

$$\begin{cases}dx_1(t)=f_x(t,x_0)\,x_1(t)\,dt+\varphi(t,x_0)\,dW(t)\\ E[Ax_1(0)+Bx_1(1)\mid\mathscr{F}_0]=0.\end{cases}\tag{4.1}$$

Its solution can be easily found from (1.3)

$$x_1(t)=e^{\int_0^t f_x(t,x_0)\,dt}\int_0^t e^{-\int_0^t f_x(t,x_0)\,dt}\varphi(t,x_0)\,dW(t).\tag{4.2}$$

And suppose $\hat\delta$ is the fixed point of S as mentioned in Section 3, then we can define a new random variable

$$\delta^*(\omega)=\begin{cases}\hat\delta, & \omega\in\Omega_{\hat\delta}\\ 0, & \omega\notin\Omega_{\hat\delta},\end{cases}\tag{4.3}$$

for which the following lemma is valid.

Lemma 3. *Under the assumptions of Theorem* 1, *the random variable* $\delta^*(\omega)$ *is* $\mathscr{F}_0$*-measurable and has the moment estimate* $E[\delta^*(\omega)]^4\leqslant K_{11}\varepsilon^4$.

Proof. The $\mathscr{F}_0$-measurability of $\delta^*(\omega)$ can be proved immediately from its definition and Theorem 2. To obtain our second assertion, it suffices to consider $E[I_{\hat\delta}\delta^*(\omega)]^4$, where $I_{\hat\delta}$ is the indicator of $\Omega_{\hat\delta}$. Since $I_{\hat\delta}\delta^*=I_{\hat\delta}\hat\delta$

and by Theorem 2 we can consider 0 to be included in the random sequence $\{\delta_n(\omega)\}$, then

$$S(\hat{\delta}) - S(0) = I_{\hat{\delta}}[T(\hat{\delta}) - T(0)] = I_{\hat{\delta}}\hat{\delta} - I_{\hat{\delta}}T(0),$$

and

$$\{E[I_{\hat{\delta}}(T(\hat{\delta}) - T(0))^4]\}^{1/4} \geqslant \{E[I_{\hat{\delta}}\hat{\delta}^4]\}^{1/4} - \{E[I_{\hat{\delta}}T^4(0)]\}^{1/4}. \quad (4.4)$$

In a way very similar to that in Theorem 2, we can find that

$$\{E(I_{\hat{\delta}}(\delta_n - \hat{\delta})^4)\}^{1/4} \to 0, \qquad \text{and} \qquad E[S(\delta_n) - S(\hat{\delta})]^4 \to 0, \text{ as } n \to \infty,$$

then from Theorem 1 we obtain

$$\begin{aligned} &\{E[I_{\hat{\delta}}(T(\hat{\delta}) - T(0))^4]\}^{1/4} \\ &\quad = \{E[S(\hat{\delta}) - S(0)]^4\}^{1/4} = \lim_{n \to \infty} \{E[S(\delta_n) - S(0)]^4\}^{1/4} \\ &\quad \leqslant \frac{1}{2} \lim_{n \to \infty} \{E[I_{\hat{\delta}}(\delta_n - 0)^4]\}^{1/4} \leqslant \frac{1}{2}\{E[I_{\hat{\delta}}\hat{\delta}^4]\}^{1/4}. \end{aligned} \quad (4.5)$$

Combining (4.5) and (4.4), we have

$$\begin{aligned} E[I_{\hat{\delta}}\hat{\delta}^4] &\leqslant K_{12}E[I_{\hat{\delta}}T^4(0)] = K_{12}E[I_{\hat{\delta}}(BJ^{-1}E(y(1,0) \mid \mathscr{F}_0))^4] \\ &\leqslant K_{13}E(y^4(1,0)) \leqslant K_{14}\varepsilon^4, \end{aligned} \quad (4.6)$$

and this completes our proof, where (2.16) and the convexity inequality of conditional expectation [12] have been used to derive the last two inequalities.

Now we can use $\delta^*(\omega)$ as an initial value of Eq. (2.4), and find its solution $y(t, \delta^*)$. If we again notice that $\Omega_{\hat{\delta}}$ has large probability for small ε, then similar to White and Franklin [10], we may consider that $x(t) = x_0(t) + y(t, \delta^*)$ is the solution of (1.5) in the meaning of large probability, or is the approximate solution of (1.5).

Furthermore we can have the expansion $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$, where $O(\varepsilon^2)$ is a random term with the moment estimation shown below.

THEOREM 3. *Under the assumptions of Theorem* 1, *for* $\varepsilon \in (0, \varepsilon_0]$ *we have*

$$E\left\{\sup_T \left[\frac{x(t) - x_0(t)}{\varepsilon} - x_1(t)\right]^2\right\} \leqslant K_{15}\varepsilon^2, \quad (4.7)$$

where K_{15} *is a constant independent on* ε.

Proof. To verify (4.7), we need the equation

$$\left\{\begin{aligned} d&\left[\frac{y(t,\delta^*)}{\varepsilon}-x_1(t)\right]\\ &=\left[f_x(t,x_0)\left(\frac{y(t,\delta^*)}{\varepsilon}-x_1(t)\right)+\frac{f_{xx}(\lambda)}{2}\left(\frac{y(t,\delta^*)}{\varepsilon}\right)^2\cdot\varepsilon\right]dt\\ &\quad+\varphi_x(\lambda)\left(\frac{y(t,\delta^*)}{\varepsilon}\right)\cdot\varepsilon\,dW(t)\\ E&\left\{A\,\frac{y(0,\delta^*)}{\varepsilon}+B\left[\frac{y(1,\delta^*)}{\varepsilon}-x_1(1)\right]\,\Big|\,\mathscr{F}_0\right\}=0,\qquad \omega\in\Omega_\delta\\ &\frac{y(0,\delta^*)}{\varepsilon}-x_1(0)=0,\qquad \omega\notin\Omega_\delta. \end{aligned}\right. \tag{4.8}$$

On Ω_δ, formula (1.3) gives the solution

$$\frac{y(t,\delta^*)}{\varepsilon}-x_1(t)=e^{\int_0^t f_x(t,x_0)\,dt}[C+I(t)],$$

where

$$I(t)=\int_0^t e^{-\int_0^t f_x(t,x_0)\,dt}\left[\frac{f_{xx}(\lambda)}{2}\left(\frac{y(t,\delta^*)}{\varepsilon}\right)^2\varepsilon\,dt+\varphi_x(\lambda)\left(\frac{y(t,\delta^*)}{\varepsilon}\right)\varepsilon\,dW(t)\right],$$

and

$$C=-BJ^{-1}E[I(1)\,e^{\int_0^1 f_x(t,x_0)\,dt}\mid\mathscr{F}_0].$$

From (2.16) and Lemma 3, we know

$$E\left\{\left[\sup_T\frac{y(t,\delta^*)}{\varepsilon}\right]^4\right\}\leqslant K_{16}, \tag{4.9}$$

from which it follows immediately that $E[\sup_T I^2(t)]\leqslant K_{17}\varepsilon^2$ and $EC^2\leqslant K_{18}\varepsilon^2$. Thus there exists a constant $K_{19}>0$, such that

$$E\left\{\sup_T\left[I_\delta\left(\frac{y(t,\delta^*)}{\varepsilon}-x_1(t)\right)^2\right]\right\}\leqslant K_{19}\varepsilon^2.$$

When $\omega\notin\Omega_\delta$, the initial value problem (4.8) has the solution

$$\frac{y(t,\delta^*)}{\varepsilon}-x_1(t)=e^{\int_0^t f_x(t,x_0)\,dt}I(t),$$

which can also be estimated by (4.9)

$$E\left\{\sup_T\left[(1-I_\delta)\left(\frac{y(t,\delta^*)}{\varepsilon}-x_1(t)\right)^2\right]\right\}\leqslant K_{20}\varepsilon^2.$$

and then this proves (4.7) immediately.

5. EXAMPLES

EXAMPLE 1. In Eq. (1.5), we suppose that $AB\geqslant 0$, $A\neq 0$, $E|\alpha^4(\omega)|<\infty$, and that $f(t,x)=A^*(t)x+a(t)$, $\varphi(t,x)=B^*(t)x+b(t)$, where A^*, B^*, a, b, are continuous functions on $[0,1]$. Then by (1.3), there exists a unique solution $x_0(t)$,

$$x_0(t)=e^{\int_0^t A^*\,dt}\left\{x_0(0)+\int_0^t a(t)\,e^{-\int_0^t A^*\,dt}\,dt\right\},$$

where

$$x_0(0)=(A+Be^{\int_0^1 A^*\,dt})^{-1}\left\{E(\alpha\mid\mathscr{F}_0)-Be^{\int_0^1 A^*\,dt}\int_0^1 a(t)\,e^{-\int_0^t A^*\,dt}\,dt\right\}.$$

Near this $x_0(t)$, we can obtain an approximate solution of $x(t)$, that is, $x(t)=x_0(t)+\varepsilon x_1(t)+O(\varepsilon^2)$, where

$$x_1(t)=e^{\int_0^t A^*\,dt}\int_0^t (B^*(t)\,x_0(t)+b(t))\,e^{-\int_0^t A^*\,dt}\,dW(t).$$

EXAMPLE 2. Now we consider a nonlinear problem,

$$dx(t)=\frac{1}{x^2+1/3}\,dt+\varepsilon x\,dW(t),\qquad x(0)+x(1)=\alpha(\omega).$$

Since $x_0(t)$ satisfies the equation

$$dx_0(t)=\frac{1}{x_0^2+1/3}\,dt,\qquad x_0(0)+x_0(1)=E(\alpha\mid\mathscr{F}_0)=\alpha_0,$$

from which we obtain

$$x_0^3(t)+x_0(t)=3t+\text{const.}\triangleq 3t+C.$$

Let $x_0(0)=\beta$, then $\beta^3+\beta=C$, which follows $x_0^3(1)+x_0(1)=3+\beta^3+\beta$, and $x_0^3(1)+x_0(1)+x_0(0)=3+\beta^3+2\beta$. So we can have $x_0^3(1)+\alpha_0=3+\beta^3+2\beta$, or

$$(\alpha_0-\beta)^3=3+\beta^3+2\beta-\alpha_0,$$

i.e.,

$$2\beta^3 - 3\alpha_0\beta^2 + \beta(2 + 3\alpha_0^2) - \alpha_0 - \alpha_0^3 + 3 = 0.$$

For every α_0, this equation has only one root β, which implies the uniqueness of $x_0(t)$. Near this $x_0(t)$, we can evaluate the approximate solution $x(t)$ easily by using the expansion formula and (4.2).

Appendix

For boundary value problem (2.1), we suppose that $x(t)$ is one of the solutions, and that $f(t, x)$ is a continuous function of $(t, x) \in [0, 1] \times (-\infty, +\infty)$, $|f(t, x)| \leqslant K_4(1 + |x|)$, and $AB \geqslant 0$, $A^2 + B^2 \neq 0$. Then we have estimate

$$|x(t)| \leqslant \begin{cases} [1 + |\beta|]\, e^{K_4(1-t)} - 1, & t \in [0, 1/2) \\ [1 + |\beta|]\, e^{K_4 t} - 1, & t \in [1/2, 1], \end{cases} \tag{A.1}$$

where $\beta = \alpha_0/(A + B)$.

Proof. Let $B = 0$, then Eq. (2.1) becomes an initial value problem. By Gronwall's inequality we know

$$|x(t)| \leqslant [1 + |x(0)|]\, e^{K_4 t} - 1 = [1 + |\beta|]\, e^{K_4 t} - 1, \qquad t \in [0, 1]. \tag{A.2}$$

Let $A = 0$, $y(T) = x(1 - T)$. We can have the equation

$$\frac{dy(T)}{dT} = -f(1 - T, y(T)), \qquad y(0) = x(1) = \frac{\alpha_0}{B} = \beta,$$

and then from (A.2), the estimations of $y(T)$ and $x(T)$,

$$\begin{aligned} |y(T)| &\leqslant [1 + |y(0)|]\, e^{K_4 T} - 1 = [1 + |x(1)|]\, e^{K_4 T} - 1, \\ |x(t)| &\leqslant [1 + |x(1)|]\, e^{K_4(1-t)} - 1 \leqslant e^{K_4(1-t)}[1 + |\beta|] - 1, \qquad t \in [0, 1]. \end{aligned} \tag{A.3}$$

If τ_1, τ_2 are two fixed numbers with $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant 1$, then by letting $u(T) = x(T + \tau_1)$ and $v(T) = x(T - 1 + \tau_2)$, we can obtain estimates similar to (A.2) and (A.3)

$$|x(t)| \leqslant [1 + |x(\tau_1)|]\, e^{K_4(t - \tau_1)} - 1, \qquad t \in [\tau_1, 1] \tag{A.4}$$

$$|x(t)| \leqslant [1 + |x(\tau_2)|]\, e^{K_4(\tau_2 - t)} - 1, \qquad t \in [0, \tau_2]. \tag{A.5}$$

Now we prove the validity of (A.1). In fact if it is incorrect, then without loss of generality, we can assume that there exists at least one point $\tau \in [0, 1/2)$, such that

$$x(\tau) > [1 + |\beta|] e^{K_4(1-\tau)} - 1. \tag{A.6}$$

Under this condition, the initial value of the corresponding solution $x(t)$ must be greater than $|\beta|$. Otherwise, by the continuity of $x(t)$, there exists at least one point $\tau_1 \in [0, \tau)$, such that

$$0 < x(\tau_1) \leqslant (1 + |\beta|) e^{K_4\tau_1} - 1 \leqslant (1 + |\beta|) e^{K_4(1-\tau)} - 1.$$

But from (A.4) we know that

$$\begin{aligned} |x(\tau)| &\leqslant [1 + |x(\tau_1)|] e^{K_4(\tau-\tau_1)} - 1 \leqslant (1 + |\beta|) e^{K_4\tau_1} e^{K_4(\tau-\tau_1)} - 1 \\ &= (1 + |\beta|) e^{K_4\tau} - 1 < (1 + |\beta|) e^{K_4(1-\tau)} - 1, \end{aligned}$$

which contradicts (A.6). Similarly we can prove that $x(1) > |\beta|$ by selecting $\tau_2 \in (\tau, 1]$ instead of τ_1. So that for the solution we must have $|Ax(0) + Bx(1)| > |A + B| \cdot |\beta| = |\alpha|$, which contradicts the boundary condition of (2.1) and then proves our result.

Remark. The condition $E \operatorname{Sup}_T x_0^4(t) < \infty$ in Lemma 1 is satisfied under the assumptions mentioned above, and another condition $|J| \geqslant K_1 > 0$ can also be satisfied if we assume $AB \geqslant 0$, $A \neq 0$, and let $K_1 = |A|$.

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